It is easily seen that if W is a closed ball of sufficiently large radius, then none of the p_t have zeros on ∂W. For

$$\frac{p_t(z)}{z^m} = 1 + t\left(a_1\frac{1}{z} + \cdots + a_m\frac{1}{z^m}\right)$$

and the term in parenthesis $\longrightarrow 0$ as $z \longrightarrow \infty$. Thus the homotopy $p_t/|p_t| : \partial W \longrightarrow S^1$ is defined for all t, so we conclude that $\deg_2(p/|p|) = \deg_2(p_0/|p_0|)$. The polynomial p_0 is just z^m; therefore every point in S^1 has precisely m preimage points in ∂W under $p_0/|p_0|$. Since we compute $\deg_2$ by counting preimages of regular values, $\deg_2(p_0/|p_0|) = m \bmod 2$. The preceding proposition now gives

One-half Fundamental Theorem of Algebra. Every complex polynomial of odd degree has a root.

This famous result was effortlessly obtained, but it also demonstrates the insufficiency of mod 2 intersection theory: too much information is wasted. The $\deg_2$ invariant is too crude to distinguish the polynomial z^2 from a constant, so it cannot prove the even half of the Fundamental Theorem of Algebra. In the next chapter we shall refine our whole approach in order to construct a theory of considerably greater power. Yet the naive methods already developed continue to produce surprisingly deep insights with a minimum of strain, as illustrated by the following two sections.

EXERCISES

1. Prove that there exists a complex number z such that

$$z^7 + \cos(|z|^2)(1 + 93z^4) = 0.$$

(For heaven's sake, don't try to compute it!)

2. Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be a sequence of smooth maps of manifolds, with X compact. Assume that g is transversal to a closed submanifold W of Z, so $g^{-1}(W)$ is a submanifold of Y. Verify that

$$I_2(f, g^{-1}(W)) = I_2(g \circ f, W).$$

(See Exercise 7, Chapter 1, Section 5.) In particular, check that if the conditions are such that one of these intersection numbers is defined, then so is the other.

3. Suppose that X and Z are compact manifolds and that $f: X \longrightarrow Y$, and $g: Z \longrightarrow Y$ are smooth maps into the manifold Y. If $\dim X + \dim Z =$

dim Y, we can define the mod 2 intersection number of f and g by $I_2(f, g) = I_2(f \times g, \Delta)$, where Δ is the diagonal of $Y \times Y$.

(a) Prove that $I_2(f, g)$ is unaltered if either f or g is varied by a homotopy.

(b) Check that $I_2(f, g) = I_2(g, f)$ [HINT: Use Exercise 2 with the "switching diffeomorphism" $(a, b) \longrightarrow (b, a)$ of $Y \times Y \longrightarrow Y \times Y$.]

(c) If Z is actually a submanifold of Y and $i: Z \hookrightarrow Y$ is its inclusion, show that

$$I_2(f, i) = I_2(f, Z).$$

(d) Prove that for two compact submanifolds X and Z in Y,

$$I_2(X, Z) = I_2(Z, X).$$

(*Note*: This is trivial when $X \pitchfork Z$. So why did we use this approach?)

*4. If $f: X \longrightarrow Y$ is homotopic to a constant map, show that $I_2(f, Z) = 0$ for all complementary dimensional closed Z in Y, except perhaps if dim $X = 0$. [HINT: Show that if dim $Z <$ dim Y, then f is homotopic to a constant $X \longrightarrow \{y\}$, where $y \notin Z$. If X is one point, for which Z will $I_2(f, Z) \neq 0$?]

*5. Prove that intersection theory is vacuous in contractable manifolds: if Y is contractable and dim $Y > 0$, then $I_2(f, Z) = 0$ for every $f: X \longrightarrow Y$, X compact and Z closed, dim X + dim Z = dim Y. (No dimension-zero anomalies here.) In particular, intersection theory is vacuous in Euclidean space.

*6. Prove that no compact manifold—other than the one-point space—is contractable. [HINT: Apply Exercise 5 to the identity map.]

*7. Prove that S^1 is not simply connected. [HINT: Consider the identity map.]

*8. (a) Let $f: S^1 \longrightarrow S^1$ be any smooth map. Prove that there exists a smooth map $g: \mathbf{R} \longrightarrow \mathbf{R}$ such that $f(\cos t, \sin t) = (\cos g(t), \sin g(t))$, and satisfying $g(2\pi) = g(0) + 2\pi q$ for some integer q. [HINT: First define g on $[0, 2\pi]$, and show that $g(2\pi) = g(0) + 2\pi q$. Now extend g by demanding $g(t + 2\pi) = g(t) + 2\pi q$.]

(b) Prove that $\deg_2(f) = q$ mod 2.

9. (Degrees are the only interesting intersection numbers on spheres.) Suppose $f: X \longrightarrow S^k$ is smooth, where X is compact and $0 <$ dim $X < k$.

Then for all closed $Z \subset S^k$ of dimension complementary to X, $I_2(X, Z) = 0$. [HINT: By Sard, there exists $p \notin f(X) \cap Z$. Use stereographic projection, plus Exercise 5.]

10. Prove that S^2 and the torus are not diffeomorphic.

11. Suppose that $f: X \longrightarrow Y$ has $\deg_2(f) \neq 0$. Prove that f is onto. [Remember that X must be compact, Y connected, and $\dim X = \dim Y$ for $\deg_2(f)$ to make sense.]

12. If Y is not compact, then $\deg_2(f) = 0$ for all maps $f: X \longrightarrow Y$ (X compact).

13. If $f: X \longrightarrow Y$ is transversal to Z and $\dim X + \dim Z = \dim Y$, then we can at least define $I_2(f, Z)$ as $\# f^{-1}(Z) \bmod 2$, as long as $f^{-1}(Z)$ is finite. Let us explore how useful this definition is without the two assumptions made in the text: X compact and Z closed. Find examples to show:
(a) $I_2(f, Z)$ may not be a homotopy invariant if Z is not closed.
(b) $I_2(f, Z)$ may not be a homotopy invariant if X is not compact.
(c) The Boundary Theorem is false without the requirement that Z be closed.
(d) The Boundary Theorem is false without the requirement that X be compact.
(e) The Boundary Theorem is false without the requirement that W be compact, even if $X = \partial W$ is compact and Z is closed. [HINT: Look at the cylinder $S^1 \times R$.]

***14.** Two compact submanifolds X and Z in Y are *cobordant* if there exists a compact manifold with boundary, W, in $Y \times I$ such that $\partial W = X \times \{0\} \cup Z \times \{1\}$. Show that if X may be deformed into Z, then X and Z are cobordant. However, the "trousers example" in Figure 2-17 shows that the converse is false.

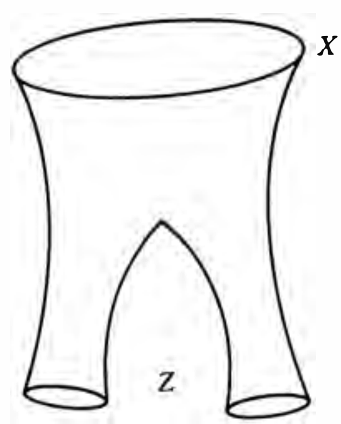

Figure 2-17

***15.** Prove that if X and Z are cobordant in Y, then for every compact manifold C in Y with dimension complementary to X and Z, $I_2(X, C) = I_2(Z, C)$. [HINT: Let f be the restriction to W of the projection map $Y \times I \longrightarrow Y$, and use the Boundary Theorem.]

16. Prove that $\deg_2(f)$ is well defined by direct application of the Boundary Theorem. [HINT: If $y_0, y_1 \in Y$, alter f homotopically to get $f \pitchfork \{y_0, y_1\}$. Now let $c: I \longrightarrow Y$ be a curve with $c(0) = y_0$, $c(1) = y_1$, and define $F: X \times I \longrightarrow Y \times Y$ by $f \times c$. Examine ∂F.]

17. Derive the Nonretraction Theorem of Section 2 from the Boundary Theorem.

18. Suppose that Z is a compact submanifold of Y with $\dim Z = \frac{1}{2} \dim Y$. Prove that if Z is globally definable by independent functions, then $I_2(Z, Z) = 0$. [HINT: By Exercise 20, Section 3, $N(Z; Y) = Z \times \mathbf{R}^k$. Certainly $I_2(Z, Z) = 0$ in $Z \times \mathbf{R}^k$, since $Z \times \{a\} \cap Z$ is empty. Now use the Tubular Neighborhood Theorem, Exercise 16, Section 3.]

19. Show that the central circle X in the open Möbius band has mod 2 intersection number $I_2(X, X) = 1$. [HINT: Show that when the ends of the strip in Figure 2-18 are glued together with a twist, X' becomes a manifold that is a deformation of X.]

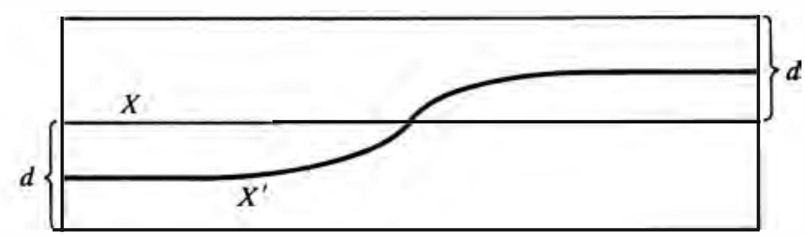

Figure 2-18

Corollary to Exercises 19 and 20. The central circle in the Möbius band is not definable by an independent function.

§5 Winding Numbers and the Jordan-Brouwer Separation Theorem

The Classical Jordan Curve Theorem says that every simple closed curve in $\mathbf{R}^2$ divides the plane into two pieces, the "inside" and "outside" of the curve. Lest the theorem appear too obvious, try your intuition on the example shown in Figure 2-19.

This section is a self-guided expedition with gun and camera into the wilds of such jungles, and in n dimensions, too! Begin with a compact, con-